FOUR LEVELS OF COGNITIVE FUNCTIONING IN ALGEBRA: AN EMPIRICAL VERIFICATION OF KUCHEMANN'S HYPOTHESIS

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A study in Queensland was undertaken to replicate a British study that examined four levels of algebraic complexity Year 8 students were capable of reaching. The levels had been identified through the British study by Kuchemann. The test items and procedures used were based on the British study. The results were similar to the British findings and indicated that a large proportion of the students had not progressed beyond the first of the four levels of cognitive functioning. The report outlines the findings and their implications within the Australian context.

A number of researchers have investigated students' difficulties and misconceptions in algebra (e.g., Booth, 1988; Pegg, 1992; Perso, 1991; Quinlan, 1992; Wagner, 1983). For example, Perso (1991) in Western Australia identified 19 of them broadly classified them into four categories as follows:

1. Understanding of variables
2. Manipulation of variables
3. Using algebraic rules to solve equations
4. Using algebraic structures/syntax to form equations

She found that the 'majority of students who have certain misconception also have the misconception which are related' (p. 10). Booth (1988) in the UK traced the students' errors in algebra to four aspects; these were as follows:

1. the focus of algebraic activities and the nature of the 'answers'

[1]The writer would like to thank Ms Dwyer of the Gold Coast for assisting with this study.

2. the use of notation and convention in algebra
3. the meaning of letters and variables
4. the kinds of relationships and methods used in arithmetic.

Perso, Booth and most of the other researchers see students' difficulties in algebra as arising partly from the nature or types of variables. Kuchemann (Hart, 1989), while admitting it to be true, has gone beyond such sources of students' difficulties and has developed the hypothesis that the students find algebra difficult because *they are unable to function cognitively at the increasing levels of complexity demanded by the various types of variables*. In fact, his exploratory study in Britain led him to hypothesize <u>four levels of cognitive functioning</u> in algebra. These levels are outlined briefly below. A fuller description can be found in Prasad (1993) and in Hart (1989, Chapter 8).

Level 1. This is the lowest level of complexity. The items at this level are purely numerical or have a simple structure and can be solved by using letters as objects, by evaluating the letter or by not using the letter at all; e.g., find x if $x = 2+4+9$. For more complex items such as $3n+4$, the students are likely to give 7 or 7n as the answer.

Level 2. The items at this level are more complex than those at Level 1 and have letters in them that have to be evaluated or used as an object. But the students still cannot cope consistently with specific unknowns, generalized numbers or variables; e.g., if $u=v+3$ and $v = 1$, then $u = ?$ Kuchemann suggested that those who progress to this level do so because of an increased familiarity with algebraic notation. In addition, his study showed that those with superior IQ did better at this level than those with lower IQ.

Level 3. This level is cognitively much more advanced than the other two levels discussed above. The questions at this level can be answered correctly by those who can cope with the use of a variable as a specific unknown for items that have a simple structure. Such students accept answers such as $2 + t$ or $x = 5y$ and regard them as meaningful.

Level 4. At this highest level of his taxonomy, the typical students are able to cope with questions that require use of a letter as specific unknowns and also have complex structures, e.g., Which is larger : g-3 or 3g? Generally, the questions at this level are complex and need the co-ordination of some operations since some limitations are attached to the value of the variable. They demand a high level of reasoning skills.

One Australian study that throws light on Kuchemann's levels was conducted in the New South Wales by Coady and Pegg (1993) using 268 university students. The authors used three[2] of Kuchemann's items taken from the third and the fourth levels. For example, "What can you say about c if $c+d = 10$ and $c < d$?' Most of the students found the three items 'very demanding in terms of cognitive functioning' (p. 191). For example, on the above item 64% of the students were wrong. An intensive analysis of the responses of a subsample of 21 students via interviews showed that the answers fell into two categories where the distinguishing feature was "the ability to take into account the 'range of possibilities and limitations' associated with relationships involving letters " (1993, p. 191). Overall, they stressed that the use of a variable in an advanced form demands cognitive functioning of a high order - one which is different from that needed when a pronumeral is interpreted as a generalized number.

Purpose of the study

The present study was conducted to verify the existence of the four levels of cognitive functioning in algebra among Queensland students and to examine some implications for curriculum and instruction in Queensland schools.

Methodology

The students who took part in this study were from one school in Queensland that was located within the Brisbane-Gold Coast corridor, an area that is rapidly developing and is attracting several new migrants from both Australia and overseas. As a Catholic high school, it had opened recently and had both boys and girls. Fifty five students from Year 8 were selected to participate in this study.

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[2] These three items were also used in the study in Queensland reported in the next section.

They took a test that was adapted from Kuchemann's test (Hart, 1989) which had been developed and validated extensively by the NFER in Britain for inclusion in the large-scale study called 'Concepts in Secondary Mathematics and Science'. Thirty items from the test were selected as follows:

Level 1: 7 items Level 2: 8 items Level 3: 8 items Level 4: 7 items.

Care was taken to exclude items that reflected the British context; in a few items the letters representing the variables were changed to offer variety for the local students. The test was administered by the mathematics teacher. Each answer was marked either right or wrong following Kuchemann. To eliminate guessing on the part of the students there were no multiple-choice items given. The test was not timed although most of the students completed it in about an hour.

Results and discussion

The findings are summarized below. The percentages shown give the mean percentage of the students who gave correct responses at a particular level.

Level		Correct (%)
1	....	61
2	....	27
3	....	11
4	....	9

On Level 1, the success rate was fairly high. It ranged from 32% to 95% with an overall mean of 61%. Clearly, the majority of students were fairly competent at this level. The questions were fairly straight-forward compared to those at the other levels and called either for numerical answers or an extremely basic operation where a pronumeral was involved but could be treated as an object.

On Level 2, the success rate (mean = 27%) was low. Some students failed to get any of the eight items correct. At the other two levels, the percentages declined even more dramatically. Only about 10% of the students were able to answer correctly the items at those levels. Many were <u>not able to answer any item</u> and left large blank spaces on their answer sheets.

To the extent that several Level 1 items reflected the curriculum at Year 8 in Queensland, the students did reasonably well at that level. A smaller number of students also succeeded at Level 2; may be because they were superior intellectually or had developed a better feel for algebraic notation - as Kuchemann had suggested with respect to his sample.

Although the present figures were slightly different from Kuchemann's figures, the overall results clearly support his findings. The overall pattern of achievement - a high success rate at Level 1 followed by a sharp decline all the way - is the same in both the studies. The somewhat higher corresponding figures in his study may suggest a somewhat better fit between his students' skills and the test since the test was based on the British mathematics curriculum. His test may also have had a higher reliability than the shorter test used in the present study.

The results from the present study also support the finding by Coady and Pegg (1993) in New South Wales. The three items used in their study were also included in the present study. None of the students in this study was able to do them correctly - a further evidence to support Coady and Pegg's (1993) finding that the letter used as a 'true' variable demands a level of cognitive functioning that is truly beyond many of the students at the high school.

Conclusion and implications

Overall, the findings from this study suggest strongly that beginning students are not able to deal with algebra that demands handling of advanced types of variables and thus provide strong empirical verification of Kuchemann's hypothesis.

Although this was a small study confined to one school, the findings strongly reinforce Kuchemann's (Hart, 1989) and Coady and Pegg's (1993) finding that beginners' difficulties in algebra are quite considerable. These difficulties are likely to be particularly severe if the curriculum and instruction did not consider the level of demand various types of variables in algebra put on the cognitive functioning of many students.

The ability to manipulate symbols alone does not indicate a high level of cognitive functioning. Rather, a high level of such functioning is indicated by the ability to take into consideration the limitations and possibilities for a variable. How to foster such an ability within their students is a real challenge to the algebra teachers.

The teaching of algebra in Queensland starts at Year 7 - the final year of primary school; none of the teachers at this level is a specialist in mathematics. Such teachers generally have a weak background of algebra and lack confidence in their ability to teach algebra. For example, while teaching a group of pre-service primary school mathematics teachers in their final year, the writer made a survey of the students' strengths and weaknesses at the beginning of the semester to help organize some remedial work during the semester. The survey revealed quite clearly that the algebraic background of the students needed strengthening. It would appear that, in teacher education programs generally, the focus on students' algebraic understanding needs increasing.

A weak grasp of variables will be a particular handicap for the students in developing problem solving skills since one of the powerful techniques of solving problems - recommended by many writers (e.g. Cockcraft, 1986) - is through casting the verbal information into an algebraic equation and then solving the equation. To the extent that students have a poor grasp of variables, they would have greater difficulties in handling such procedures for problem solving or in developing a structural conception of algebra as opposed to the procedural. Hence, one way of improving the students' ability to solve problems is to give them a better understanding of algebra - particularly of advanced types of variables.

The study discussed here was a small one restricted to one school and therefore should be regarded as a pilot study. A larger study with focus on senior students in Queensland is being planned.

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